A non-archimedean analogue of Levine formula

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Poincaré-Lelong and Green currents

Proposition (Poincaré-Lelong).

Let X be a complex manifold, $\overline{L}$ a hermitian line bundle on X and s a regular meromorphic section of L. Then

$$\mathrm{dd^c}[-\log\|s\|^2]+\delta_{\mathrm{div}\, s}=[c_1(\overline{L})].$$

- Equality between **currents**, *ie* linear functionals $A(X)\to\mathbb{C}$, with $A(X)$ the space of differential forms. Currents generalize distributions in higher dimension.
- If $Z\subseteq X$ is a submanifold, define $\delta_Z(\alpha):=\int_Z\alpha$. We extend the definition by linearity.
- If ω is a differential form, define $[\omega](\alpha):=\int_X\omega\wedge\alpha$.
- The differential operator $\mathrm{dd^c}$ is defined on currents by duality.

Definition. Let Z be a cycle on X. A **Green current** for Z is a current g such that there exists a differential form ω satisfying

$$\mathrm{dd^c}g+\delta_Z=[\omega].$$

Green currents are of uttermost importance in the construction of **arithmetic Chow groups** [4]

Levine formula

Let X and $\overline{L}$ as above, $r\geq 1$, and $s_1,\dots,s_r$ sections of L. Assume that the sections form a **regular sequence**. Then $Z:=\{s_1=\cdots=s_r=0\}$ is a closed subset of pure codimension r. Denote $\tau=\log(\sum_k\|s_k\|^2)$, $\omega=c_1(\overline{L})$, and $\omega_0=\omega+\mathrm{dd^c}\tau$.

Theorem (Levine [6, 5])

Let
$$\Lambda=-\tau\wedge\sum_{k=0}^{r-1}\omega^{r-1-k}\wedge\omega_0^k,$$
then
$$\mathrm{dd^c}[\Lambda]+\delta_Z=[\omega^r].$$

Non-archimedean Green currents: why?

- Arithmetic Chow groups mix algebraic objects at finite places (intersection on schemes) and analytic objects at infinite places (Green currents): **asymmetry**.
- Similar in nature to the product formula: if $f=\prod_p p^{n_p}\in\mathbb{Q}^*$,

$$\log|f|-\sum_p n_p\log p=0\quad\text{ie}\quad |f|\prod_p|f|_p=1.$$

The pink term is analytic, while orange terms come from algebra. But they have a **non-archimedean analytic interpratation** !

Question. Can we look at the objects involved in arithmetic intersection theory in a more unified way by constructing **non-archimedean Green currents** ?

Bloch–Gillet–Soulé formalism

Let R a discrete valuation ring, $K=\operatorname{Frac}R$ and X a smooth K-variety.

- A model of X is a regular proper flat R-scheme $\mathfrak{X}$, whose generic fiber if isomorphic to X.

Bloch, Gillet and Soulé [1] propose definitions of forms and currents on X, by doing constructions from intersection theory on the special fibers of **all models** of X.

- For example, a closed differential form in $A^{k,k}(X)$ is defined to be an element of $\operatorname{colim}\mathrm{CH}^k(\mathfrak{X}_0)$, the colimit ranging over all models, with respect to pullback maps.
- For $Z\subseteq X$, the current of integration δ_Z is the collection of the closures of Z in the models.
- The analogue in this setup of $-\log\|s\|^2$ is a **current** denoted $\mathrm{div_v}(s)$. On a model $\mathfrak{X}$ it is given by the vertical part of the divisor of s on $\mathfrak{X}$.

They also define notions of metrics and Chern forms, and they prove a counterpart of the Poincaré-Lelong equation when $L=\mathcal{O}_X$.

Our results

Theorem ([3])

Let X a smooth K-variety, L a line bundle on X, $s_1,\dots,s_r$ a regular sequence of sections of L. Let $\pi\colon\widetilde{X}\to X$ be the blow-up of X along $Z:=\{s_1=\cdots=s_r=0\}$, E the exceptional divisor, s the canonical section of $O_{\widetilde{X}}(E)$ and q the second projection $\widetilde{X}\to\mathbb{P}_K^{r-1}$. Denote $\alpha=\pi^*c_1(\overline{L})$, $\beta=q^*c_1(\overline{\mathcal{O}_{\mathbb{P}_K^{r-1}}(1)})$ and

$$\Lambda=\pi_*\left(\mathrm{div_v}(s)\wedge\sum_{k=0}^{r-1}\alpha^{r-1-k}\wedge\beta^k\right).\qquad\text{Then}\quad \mathrm{dd^c}\Lambda+\delta_Z=[\alpha^r].$$

- Constructions **on the blow-up** because we can only define objects on the whole of X in BGS setup.
- This is justified, because similar constructions in the archimedean setup give back Levine formula exactly.
- We have an explicit analog of $\log\sum\|s_i\|^2$, constructed on X itself and not on the blow-up $\widetilde{X}$.
- We also have a generalization when $Z=\{\sigma=0\}$, with σ a **regular section of a vector bundle**. This is the equivalent of an archimedean result from [2].

References

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